Asymptotic Statistics : Asymptotic Normality of Z Estimators

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Setup

Context: Z-estimator $\hat{\theta}_n$ s.t.

$$Z_n(\hat{\theta}_n) = \frac{1}{n} \sum_{i=1}^{n} z(X_i, \hat{\theta}_n) = 0$$

with $X_1, \ldots$ i.i.d $\qquad z : \mathbb{R}^p \times \Theta \to \mathbb{R}$

$$\exists \theta_0 \in \Theta \text{ s.t. } \mathbb{E}\left(z(X_1, \theta_0)\right) = 0$$

$$\hat{\theta}_n \xrightarrow[n \to +\infty]{\mathbb{P}} \theta_0 \quad \text{(cf lectures 4 and 5)}.$$

Intuition

Taylor around θ_0: (which makes sense since $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta_0$)

$$0 = Z_n(\hat{\theta}_n) \approx Z_n(\theta_0) + (J Z_n)(\theta_0)(\hat{\theta}_n - \theta_0).$$

" $\Rightarrow$ " $\quad \sqrt{n}(\hat{\theta}_n - \theta_0) = -(J Z_n)(\theta_0)^{-1}(\sqrt{n}\, Z_n(\theta_0))$

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L.L.N $\quad (J Z_n)(\theta_0) \xrightarrow{\mathbb{P}} \mathbb{E}\left(JZ(\theta_1, \theta_0)\right) =: J$

and if $\det J \neq 0$, C.M $\Rightarrow (JZ_n)(\theta_0)^{-1} \xrightarrow{\mathbb{P}} J^{-1}$

C.L.T. $\quad \sqrt{n}\left[Z_n(\theta_0) - \underbrace{\mathbb{E}[Z(X_1, \theta_0)]}_{=0}\right] \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \mathrm{cov}(Z(X_1, \theta_0))\right)$

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Slutsky

$$\sqrt{n}\,(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, J^{-1} \operatorname{cov}\left(Z(X_1, \theta_0)\right) J^{-T}\right)$$

Objective of this lecture: Make this rigorous.

Maximal Inequality

<u>Theorem (Maximal Inequality)</u>

- $(X_i)_{i \geq 1}$ iid with distribution $\mathcal{L}(\mathbb{R}^d)$
- $\mathcal{F}$: set of functions from $\mathbb{R}^k$ to $\mathbb{R}$.
- $\exists\, F : \mathbb{R}^k \to \mathbb{R}$ s.t. $\forall f \in \mathcal{F}$, $|f(\cdot)| \leq F(\cdot)$ $\mathcal{L}$-a.e., $C_F := \mathbb{E}\left(F(X_1)^2\right) < \infty$.

<u>Then</u> $$\mathbb{E}^* \sup_{f \in \mathcal{F}} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_1)) \right| \leq C_{MI} \int_0^{\sqrt{C_F}} \sqrt{\log\left(N_{[\,]}(F, L^2(\mathcal{L}), \varepsilon)\right)}\, d\varepsilon$$

Maximal Inequality

Exercice If $X_1, \dots, \overset{i.i.d}{\sim} \text{Unif}([0,1])$, $\mathcal{F} = \{\mathbb{1}_{(-\infty, t]} ; t \in [0,1]\}$,

find an upper-bound on

$$\underbrace{\mathbb{E}^* \sup_{f \in \mathcal{F}} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_i)) \right|}_{*}.$$

Application to Z Estimators

Theorem (Asymptotic Normality of Z-estimators). If,

- $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} z(X_i, \hat{\theta}_n) = o_{\mathbb{P}}(1)$ with $z : \mathbb{R}^k \times \Theta \to \mathbb{R}^p$ st
  $\mathbb{E}(\|z(X_1, \theta)\|^2) < \infty \ \forall \theta$,

  And $\exists \theta_0 \in \mathring{\Theta}$ st $\mathbb{E}(z(X_1, \theta_0)) = 0$,
  $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta_0$.

- $\exists \mathcal{N}$ neighborhood of θ_0 s.t. $\theta \mapsto \mathbb{E}(z(X_1, \theta)) \in C^1(\mathcal{N})$.
  $J\mathbb{E}(X_1, \theta_0)$ is invertible.

- $\mathcal{F}_j := \{ x \mapsto z(x, \theta)_j \ ; \theta \in A \}$.
  $\forall j, \forall \delta > 0, \int_0^\delta \sqrt{\log \mathcal{N}_{[\,]}(\mathcal{F}_j, L^2(\mathcal{L}), \varepsilon)} \, d\varepsilon < \infty$

- $\mathbb{E} \sup_{\substack{\theta \in \mathcal{N} \\ \|\theta - \theta_0\| \leq \delta}} \|z(X_1, \theta) - z(X_1, \theta_0)\|^2 \xrightarrow[\delta \to 0]{} 0$.

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<u>Theorem (Asymptotic Normality of Z-estimators).</u> <u>If,</u>

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<u>Then</u>

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) = -\left(J\mathbb{E}\left(z(X_1, \theta_0)\right)\right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z(X_i, \theta_0) + o_{\mathbb{P}}(1)$$

and thus

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow[n \to +\infty]{\mathcal{L}} \mathcal{N}\left(0, J\mathbb{E}\left(z(X_1, \theta_0)\right)^{-1} \operatorname{cov}\left(z(X_1, \theta_0)\right) J\mathbb{E}\left(z(X_1, \theta_0)\right)^{-T}\right).$$

Asymptotic Normality of the Median

$(X_i)_{i \in \mathbb{N}}$ iid with F_{X_1} and with density f

Assumption 1: $f(x) > 0 \quad \forall x \in \mathbb{R}$ almost surely.

Population median: θ_0 the only real number such that $F_{X_1}(\theta_0) = \frac{1}{2}$

Empirical median: $\hat{\theta}_n$ s.t. $\sum_{i=1}^{n} \operatorname{sgn}(\hat{\theta}_n - X_i) = 0$

Remark: We might restrict n to even numbers and that all the X's are different (it happens almost surely).

$\Rightarrow$ By lecture 4, $\theta_n \xrightarrow{\mathbb{P}} \theta_0$ (The first point of the theorem is satisfied)

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Assumption 2: f is continuous on a neighborhood of θ_0.

$\Rightarrow \mathbb{E}\left(\mathrm{sgn}(\theta - X_1)\right) = 2F_{X_1}(\theta) - 1$ is C^1 on a neighborhood of θ_0 with derivatives $2f(\theta_0)$ at θ_0.

(The second point of the theorem is satisfied).

Next Step: $\mathcal{F} := \{x \mapsto \mathrm{sgn}(\theta - x) ; \theta \in \mathbb{R}\}$

Control $\mathcal{N}_{[\,]}(\mathcal{F}, L^2(\mathcal{L}), \varepsilon)$

Asymptotic Normality of the Median

Excercice :

1) Show that $\forall N$ st $N+1 \geq \frac{1}{\varepsilon^2}$, $\exists t_1, \ldots, t_N$ st

$\forall j = 0, \ldots, N$, $\mathcal{L}\left((t_j, t_{j+1})\right) \leq \varepsilon^2$

with $t_0 = -\infty$ and $t_{N+1} = +\infty$.

2) By considering functions of the form $x \mapsto \mathbb{1}(t_j \leq x)$ and $x \mapsto \mathbb{1}(t_j < x)$,

Build brackets of $\mathcal{F}_- = \{x \mapsto \mathbb{1}(\theta < x) ; \theta \in \mathbb{R}\}$.

Do the same thing for $\mathcal{F}_+ = \{x \mapsto \mathbb{1}(x < \theta) ; \theta \in \mathbb{R}\}$

3) After noticing that $\mathrm{sgn}(\theta - x) = \mathbb{1}(x < \theta) - \mathbb{1}(\theta < x)$, conclude.