# Asymptotic Statistics : Bracketing Numbers and Consistency

## M2RI - Toulouse University

**Clément Lalanne**

# Uniform Convergence in Probability

<u>Context</u> With M and Z estimators, we need to show

$$\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \sup_{\theta \in \Theta} \|Z_n(\theta) - Z(\theta)\| \xrightarrow{\mathbb{P}} 0 .$$

This lecture will explain how to do this for M-estimators, Z-estimators being handled in the same way.

# Uniform Convergence in Probability

Hypothesis: • $X_1, \dots$ iid

• $M_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} m(X_i, \theta)$ $\quad m : \mathbb{R}^k \times \Theta \to \mathbb{R}$

• $M(\theta) = \mathbb{E}(m(X_1, \theta))$

Notation: $\mathcal{F} = \{ m(\cdot, \theta) ; \theta \in \Theta \}$.

Rewriting: $\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| = \sup_{f \in \mathcal{F}} |M_n(f) - M(f)| = \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_1)) \right|$

# Measure of Complexity

: We need to measure the <u>complexity</u> of the class of functions $F$.

# Covering Numbers ?

1st Idea Since we know how to handle $F'$ finite (Law of large numbers).

So we can search $F'$ finite such that $\forall f \in F$, $\exists f' \in F'$ s.t. $\|f - f'\|_\infty \leq \varepsilon$.

Then $$\sup_{f \in F} \left| \frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mathbb{E}\left[f(X_1)\right] \right| \leq \sup_{f' \in F'} \left| \frac{1}{n}\sum_{i=1}^{n} f'(X_i) - \mathbb{E} f'(X_1) \right| + 2\varepsilon$$

p.s.

But $N(F, \|\cdot\|_\infty, \varepsilon) = \inf_{N \in \mathbb{N}} \{N : \exists F' \text{ with } |F'| = N \text{ and } \forall f \in F, \exists f' \in F', \|f - f'\|_\infty \leq \varepsilon\}$

(covering number) can be $\infty$ on simple examples.

Exercice What is $N(F, \|\cdot\|_\infty, \varepsilon)$ when

$$F = \{ \mathbb{1}_{(-\infty, t]} \, ; \, t \in [0,1] \} \quad ?$$

# Bracketing Numbers

Definition (Bracketing number)

For $\ell$ and $u$ two functions from $\mathbb{R}^k$ to $\mathbb{R}$ and such that $\forall x,\ \ell(x) \leq u(x)$.

$$[\ell, u] := \left\{ f : \mathbb{R}^k \to \mathbb{R} \ : \ \forall x,\ \ell(x) \leq f(x) \leq u(x) \right\} \text{ (bracket)}.$$

We define the <u>bracketing number</u> $\mathcal{N}_{[\,]}(\mathcal{F}, L^q(\mathcal{L}), \varepsilon)$ as

$$\mathcal{N}_{[\,]}(\mathcal{F}, L^q(\mathcal{L}), \varepsilon) = \inf_{N \in \mathbb{N}} \left\{ \exists [\ell_1, u_1], \dots, [\ell_N, u_N] : \forall j,\ \|u_j - \ell_j\|_{L^q(\mathcal{L})} \leq \varepsilon,\ \mathcal{F} \subset \bigcup_j [\ell_j, u_j] \right\}.$$

# Bracketing Numbers

Exercice $P = \mathrm{Unif}[0,1]$ $\mathcal{F} = \left\{ f_t(x) = \mathbb{1}_{(-\infty, t]}(x) : t \in [0,1] \right\}$

Find (up to mult. constants) $N_{[\,]}\left(\mathcal{F}, L_2(P), \varepsilon\right)$.

# Glivenko-Cantelli Families

Definition ($\mathcal{L}$-Glivenko-Cantelli)

$F$ is said $\mathcal{L}$-Glivenko-Cantelli if

- $\forall f \in F$, $\int |f| d\mathcal{L} < \infty$
- $\forall (X_i)_{i \in \mathbb{N}}$ i.i.d with distribution $\mathcal{L}$,

$$\sup_{f \in F} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E}\left[f(X_1)\right] \right| = o_{\mathbb{P}}(1).$$

# Glivenko-Cantelli Families

Theorem: If $\mathcal{F}$ is such that

- $\forall f \in \mathcal{F}$, $\int |f| \, d\mathcal{L} < \infty$
- $\forall \varepsilon > 0$, $\mathcal{N}_{[\,]}(\mathcal{F}, L^1(\mathcal{L}), \varepsilon) < \infty$.

Then $\mathcal{F}$ is $\mathcal{L}$-Glivenko-Cantelli.

# Consistency on Compact Spaces

**Proposition** Let $\mathcal{L}$ be a distribution on $\mathbb{R}^k$, $F = \{g_\theta ; \theta \in \Theta\}$ with $\forall \theta$, $g_\theta : \mathbb{R}^k \to \mathbb{R}$. Furthermore, let's assume that

- $\Theta$ is a compact set of a metric space
- $\forall x$, $\theta \mapsto g_\theta(x)$ is continuous
- $\int \sup_{\theta \in \Theta} |g_\theta(x)| \, d\mathcal{L}(x) < \infty$.

Then $F$ is $\mathcal{L}$-Glivenko-Cantelli.

# Consistency of Maximum Likelihood

<u>Theorem</u> Statistical Model $\{\mathcal{L}_\theta \,;\, \theta \in \Theta\}$ with $\forall \theta$, $\mathcal{L}_\theta$ with density $f_\theta$.

$X_1, \dots \overset{iid}{\sim} f_{\theta_0}$ , $\hat{\theta}_n$ Maximum Likelihood Estimator.

<u>Assumptions:</u>
- $\Theta$ Compact
- $\forall \theta$, $f_\theta(x) > 0$ a.s.
- $\forall x$, $\theta \mapsto f_\theta(x)$ is continuous
- $\int \sup_{\theta \in \Theta} |\log(f_\theta(x))| \, f_{\theta_0}(x)\, dx < \infty$
- $\theta \mapsto \mathcal{L}_\theta$ is injective on $\Theta$.

<u>Then</u> : $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta_0$.