Asymptotic Statistics : Delta Method and Method of Moments

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Delta Method : Motivations

Setup: We have $(\hat{\theta}_n)$ a sequence of estimators for θ and a mapping ϕ.

Question: Is $(\phi(\hat{\theta}_n))$ a good sequence for estimating $\phi(\theta)$?

Simple Example: if $\theta_n \xrightarrow[n \to +\infty]{\mathbb{P}} \theta$ and ϕ is continuous at θ, then $\phi(\theta_n) \xrightarrow[n \to +\infty]{\mathbb{P}} \phi(\theta)$.

Question: But what if we consider a stronger result like asymptotic normality ?

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow[n \to \infty]{\mathscr{L}} \mathcal{N}(0,\Sigma)$$

Delta Method

Simple Case: For linear functions, asymptotic normality is preserved.

$$\sqrt{n}\left(M\hat{\theta}_n - M\theta\right) \xrightarrow[n\to\infty]{\mathcal{L}} M\mathcal{N}(0,\Sigma) = \mathcal{N}(0,M\Sigma M^T)$$

Theorem (Delta Method): Let $\theta \in \mathbb{R}^k$, $\phi : \mathbb{R}^k \to \mathbb{R}^m$ differentiable at θ, $(\hat{\theta}_n)$ be a sequence of random vectors, X be a random vector and (r_n) be a sequence of real numbers that fors to infinity.

If

$$r_n\left(\hat{\theta}_n - \theta\right) \xrightarrow[n\to\infty]{\mathcal{L}} X$$

Then

$$r_n\left(\phi(\hat{\theta}_n) - \phi(\theta)\right) \xrightarrow[n\to\infty]{\mathcal{L}} (J\phi(\theta))X \text{ and } r_n\left(\phi(\hat{\theta}_n) - r_n(J\phi(\theta))(\hat{\theta}_n - \theta)\right) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Delta Method: Example for Variance Estimation

Let $X_1, X_2, \ldots, X_n, \ldots \overset{\text{i.i.d.}}{\sim} X$ with $\mathbb{E}(|X|^4) < \infty$

Denoting by $\hat{\mu}_{2,n} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ the empirical variance, use the delta method to show its asymptotic normality as an estimator of the variance.

Hint: One can notice that

$$\hat{\mu}_{2,n} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{1}{n}\sum_{i=1}^{n}X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X))^2 - \left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X))\right)^2$$

Method of Moments

$\{\mathscr{L}_\theta; \theta \in \Theta\}$a statistical model, $\Theta \subset \mathbb{R}^d$, there exists $\theta_0 \in \Theta$ such that $X_1, X_2, \ldots, X_n, \ldots \overset{\text{i.i.d.}}{\sim} \mathscr{L}_{\theta_0}$

Method of moments: Given $f_1, \ldots, f_p : \mathbb{R}^d \to \mathbb{R}$, build an estimator $\hat{\theta}$ that solves

$$\begin{cases} \dfrac{1}{n}\sum_{i=1}^{n} f_1(X_i) = \mathbb{E}_{\hat{\theta}}\big[f_1(X_1)\big] \\ \vdots \\ \dfrac{1}{n}\sum_{i=1}^{n} f_p(X_i) = \mathbb{E}_{\hat{\theta}}\big[f_p(X_1)\big] \end{cases}$$

Exercice: Let $\Theta = \mathbb{R} \times (0,\infty)$, $\theta = (m, \sigma^2)$ and $\mathscr{L}_\theta = \mathscr{N}(m, \sigma^2)$. Let us consider the method of moments with $f_1(x) = x$ and $f_2(x) = x^2$. Find the corresponding estimator.

Method of Moments: Asymptotic Normality

Theorem: Let us define the function $e : \Theta \to \mathbb{R}^p$ by

$$e(\theta) = \begin{pmatrix} \mathbb{E}_\theta[f_1(X_1)] \\ \vdots \\ \mathbb{E}_\theta[f_p(X_1)] \end{pmatrix}.$$

Assume that $\theta_0 \in \dot{\Theta}$ and there exists $\varepsilon > 0$ such that $B(\theta_0, \varepsilon) \subset \Theta$ and such that e is continuously

differentiable on $B(\theta_0, \varepsilon)$ with an invertible Jacobian matrix $Je(\theta_0)$ at θ_0. Assume also that for $j = 1,\ldots,p$, $\mathbb{E}\left[\,|f_j(X_1)|^2\,\right] < \infty$. Then we can define a random vector $\hat{\theta}_n$ that is a solution of the method of moments with probability going to 1, and

$$\sqrt{n}\,(\hat{\theta}_n - \theta_0) \xrightarrow{\mathscr{L}} \mathcal{N}\left(0, (Je(\theta_0))^{-1}\Sigma_f(Je(\theta_0))^{-T}\right),$$

where Σ_f is the covariance matrix of the random vector $(f_1(X_1), \ldots, f_p(X_1))$.